Solve this recurrence: $a_0 = 1,\ a_1 = 2$

$a_n = a_{n-1} + 2a_{n-2} + 2^n + 2n,$

Solution: ① Homogeneous Solution

$$a_n = a_{n-1} + 2a_{n-2}$$

Try $a_n = r^n \Rightarrow r^n = r^{n-1} + 2r^{n-2}$

$\Rightarrow r^n - r^{n-1} - 2r^{n-2} = 0 \underset{\text{divide } r^{n-2}}{\Rightarrow} r^2 - r - 2 = 0$

$(r-2)(r+1) \quad r = 2 \text{ or } -1.$

Gen homogeneous Solution: $a_n = C_1 2^n + C_2(-1)^n.$

② Find particular solution – any solution

Want $a_n =$ something

$$a_n = a_{n-1} + 2a_{n-2} + 2^n + 2n.$$

Try $a_n = k_1 2^n + k_2 n + k_3$ ☆

$LHS = k_1 2^n + k_2 n + k_3$

$= RHS = (k_1 2^{n-1} + k_2(n-1) + k_3) + 2(k_1 2^{n-2} + k_2(n-2) + k_3)$

$+ 2^n + 2n$

$\Leftrightarrow k_1 2^n + k_2 n + k_3$

$= \left(\frac{k_1}{2} + \frac{2k_1}{4} + 1\right) 2^n + (k_2 + 2k_2 + 2)n$

$+ (-k_2 + k_3 - 4k_2 + 2k_3)$

$k_1 2^n + k_2 n + k_3 = (k_1 + 1)2^n + (3k_2 + 2)n$

$+ (-5k_2 + 3k_3)$

$k_1 = k_1 + 1$

OK $\begin{cases} k_2 = 3k_2 + 2 \\ k_3 = -5k_2 + 3k_3 \end{cases}$

This won't work — there is no particular solution of that form

Next attempt:

Now let's try

$$a_n = k_1 n 2^n + k_2 2^n + k_3 n + k_4$$

We plug in:

$$a_n = a_{n-1} + 2a_{n-2} + 2^n + 2n.$$

$$(k_1 n 2^n + k_2 2^n + k_3 n + k_4) = k_1(n-1)2^{n-1} + k_2 2^{n-1} + k_3(n-1) + k_4$$
$$+ 2\left[k_1(n-2)2^{n-2} + k_2 2^{n-2} + k_3(n-2) + k_4\right]$$
$$+ 2^n + 2n$$

$$\text{RHS} = \left(\frac{k_1}{2} + \frac{2k_1}{4}\right)n2^n + \left(\overbrace{\frac{-k_1}{2} + \frac{k_2}{2} - \frac{4k_1}{4} + \frac{2k_2}{4} + 1}^{-\frac{3}{2}k_1 + k_2 + 1}\right)2^n$$
$$+ (k_3 + 2k_3 + 2)n + (\underbrace{-k_3 + k_4 - 4k_3 + 2k_4}_{-5k_3 + 3k_4})$$

4 equations

$$k_1 = \frac{k_1}{2} + \frac{2k_1}{4} \qquad k_1 = k_1 \checkmark \qquad (n2^n \text{ term})$$
$$k_2 = -\frac{3}{2}k_1 + k_2 + 1$$

$$k_3 = \cancel{k_3 + 2k_3 + 2} \quad 3k_3 + 2 \qquad (n \text{ term})$$

$$k_4 = -5k_3 + 3k_4 \qquad (\text{const term})$$

$$k_2 = \text{anything} \overset{\text{set}}{=} 0$$
$$\frac{3}{2}k_1 = 1 \Rightarrow k_1 = \frac{2}{3}$$

$$-2k_3 = 2 \Rightarrow k_3 = -1$$
$$k_4 = 5 + 3k_4 \Rightarrow -2k_4 = 5 \Rightarrow k_4 = -\frac{5}{2}$$

particular solution is

$$\boxed{a_n^{part} = \frac{2}{3} n 2^n - n - \frac{5}{2}}$$

$\Rightarrow$ General Solution to inhomogeneous formula is

$$\boxed{a_n = C_1 2^n + C_2 (-1)^n + \frac{2}{3} n 2^n - n - \frac{5}{2}}$$

Plug in initial conditions

$$a_0 = 1 = C_1 + C_2 + 0 - 0 - \frac{5}{2}$$

$$a_1 = 2 = 2C_1 - C_2 + \frac{4}{3} - 1 - \frac{5}{2}$$

$$\Rightarrow C_1 + C_2 = \frac{7}{2}$$

$$+ 2C_1 - C_2 = \cancel{2+1}\,3 - \frac{4}{3} + \frac{5}{2}$$

$$\frac{18 - 8 + 15}{6} = \frac{25}{6}$$

$$C_1 + C_2 = \frac{7}{2} = \frac{21}{6}$$

$$+ 2C_1 - C_2 = \frac{25}{6}$$

add $3C_1 = \frac{46}{6} = \frac{23}{3}$

$$\Rightarrow C_1 = \frac{23}{9}$$

$$+ \frac{23}{9} + C_2 = \frac{21}{6}$$

$$C_2 = \frac{21}{6} - \frac{23}{9} = \frac{63 - 46}{18}$$

$$= \frac{17}{18}$$

Our solution:

$$\boxed{a_n = \frac{23}{9}\cdot 2^n + \frac{17}{18}(-1)^n + \frac{2}{3}n2^n - n - \frac{5}{2}}$$

More General Inclusion-Exclusion.

★ If $S = S_1 \cup S_2$ are sets

$$|S| = |S_1| + |S_2| - |S_1 \cap S_2|$$

If $S = S_1 \cup S_2 \cup S_3$ are sets

$$|S| = |S_1| + |S_2| + |S_3| - |S_1 \cap S_2| - |S_2 \cap S_3| - |S_1 \cap S_3| + |S_1 \cap S_2 \cap S_3|$$